On DRC-Covering of K_n by Cycles

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N° 4299

Octobre 2001

THÈME 1

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--4299--FR+ENG

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Thème 1 — Réseaux et systèmes
Projet MASCOTTE

Rapport de recherche n° 4299 — Octobre 2001 — 16 pages

Abstract: This work considers the cycle covering of complete graphs motivated by the design of survivable WDM networks, where the requests are routed on sub-networks which are protected independently from each other. The problem can be stated as follows: for a given graph G, find a cycle covering of the edge set of K_n, where $V(K_n) = V(G)$, such that each cycle in the covering satisfies the disjoint routing constraint (DRC), relatively to G, which can be stated as follows : to each edge of K_n we associate in G a path and all the paths associated to the edges of a cycle of the covering must be vertex disjoint. Here we consider the case where $G = C_n$, a ring of size n and we want to minimize the number of cycles in the covering. We give optimal solutions for the problem and as well as for variations of the problem, namely, its directed version and the case when cycle length is fixed as 4.

Key-words: cycle covering, WDM, graph, survivability

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A propos de la DRC-couverture de K_n par des cycles

Résumé : Dans ce travail, nous étudions la couverture du graphe complet par des cycles, motivé par le dimensionnement de réseaux WDM tolérants aux pannes, où les requêtes sont routées sur des sous-réseaux, protégés indépendement les uns des autres.

Le problème général peut être exprimé de la façon suivante: étant donnés un graphe G modélisant un réseau WDM et un graphe logique I modélisant une instance de requêtes sur G, trouver une couverture par des cycles des arêtes de I, telle que chaque cycle de la couverture satisfasse la contrainte de routage disjoint (DRC) sur G. La DRC s'exprime de la façon suivante: associer à chaque arête de I un chemin dans G tel que tous les chemins associés aux arêtes d'un cycle de la couverture soient sommets disjoints.

Ici, nous considérons le cas où $G = C_n$ (un cycle de longueur n) et $I = K_n$ (l'instance all-to-all). Nous cherchons à minimiser le nombre de cycles de la couverture. Nous donnons une solution optimale à ce problème ainsi qu'au cas où la longueur des cycles de la couverture est fixée à 4, et aux versions orientés de ces problèmes.

Mots-clés : couverture par des cycles, WDM, graphe, tolérance aux pannes

1 Introduction

The problem of covering the edges of K_n by complete graphs K_k's has been studied by many people. This problem is known as the covering design problem [8, 10]. Moreover, the problem of finding a perfect covering of the edges of K_n is the same as a partitioning problem and this is related to the existence of an $(n, k, 1)$-design. Partitioning the edges of K_n into isomorphic graphs and in particular cycles C_k's has also been well studied [6, 7]. But there seems to be less known results for the problem of covering the edges of K_n by C_k's, where $k \geq 4$. In [2], the answer was given for $k = 4$.

Here we consider a covering problem arising from the design of a survivable WDM network G, where the communication requests are routed on sub-networks which are protected independently from each other. Designers prefer to use small cycles as subnetworks because the failure of one edge is easily repaired by using the remaining part of the cycle. This problem was asked by France Telecom R & D (see [3] for more details on it).

Routing a request over G consists in finding a path over G between the pair of nodes communicating in the request. We cover the requests by cycles and each cycle formed by some requests must be routed vertex disjointly over G, or this is the same as saying that we can find a set of vertex disjoint paths corresponding to the set of requests over a cycle. We call this property the disjoint routing constraint (DRC). This property can be extended to the directed version of the problem in which the paths in the routing will be directed paths and the cycles in the covering will be directed cycles.

It is clear that not every cycle satisfies DRC. As an illustration, the cycle $(1, 2, 3, 4, 1)$ can be routed over C_4 and it satisfies DRC. However the cycle $(1, 3, 4, 2, 1)$ does not satisfy DRC as it is impossible to associate the requests $(1, 3)$ and $(2, 4)$ to vertex disjoint paths in C_4(see Figure 1).

In general, we can model all the requests as the edge set of a logical graph I undirected or not. Finding a cycle covering for all requests with the above constraint is the same as finding a covering of the edge set of I, such that each cycle in the covering satisfies the DRC constraint in G. Such a covering will be called a DRC-covering of I relatively to G.

Our aim is to minimize the cost of the network ; the cost function is a complicated one, but for the particular case, considered here, where $G = C_n$, it corresponds to minimize the number of cycles in the covering.

Finally we will suppose that there is a request of communication between every pair (or couple) of vertices . Hence, I will be the complete graph K_n or the symmetric complete digraph K_n^*.

As an illustration, let G be C_4 and I be K_4 (See Figure 1). A first covering is given by the two C_4's $(1, 2, 3, 4, 1)$ and $(1, 3, 4, 2, 1)$ (See Figure 1.(c)), but there does not exist an edge disjoint routing for the cycle $(1, 3, 4, 2, 1)$. In counterpart, the covering given in Figure 1.(d) by the C_4 $(1, 2, 3, 4, 1)$ and the two C_3's $(1, 2, 4, 1)$ and $(1, 3, 4, 1)$, satisfy the edge disjoint routing property.

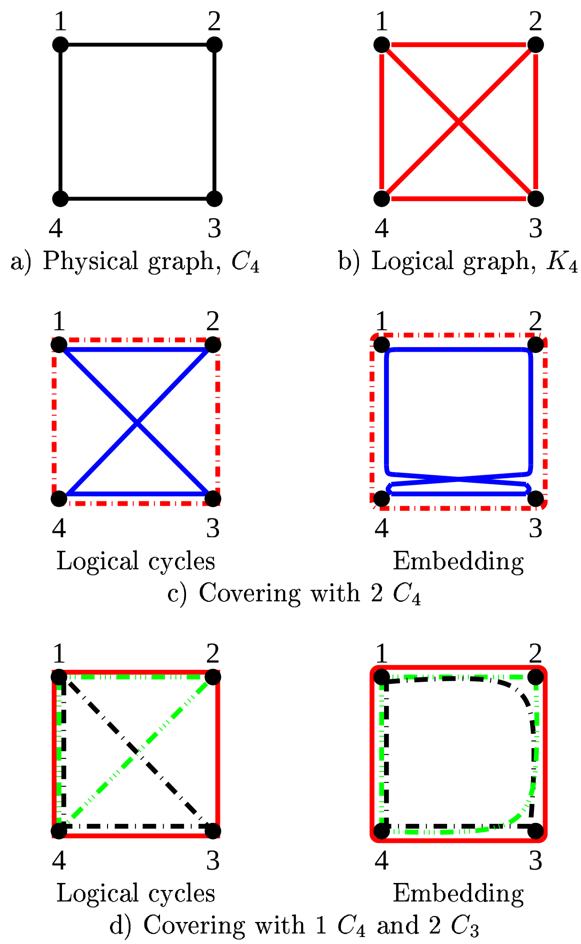

Figure 1: Cycle covering example.

In summary, we want to find the minimum number of cycles in a DRC-covering of K_n (or K_n^*) relatively to the cycle C_n (or $\vec{C}_n$). As a variation of the problem, we can also add some restriction to the cycles in the covering, for example, we can consider the case when the size of the cycles is uniform or is bounded.

We denote $\rho(n)$ the minimum number of cycles needed in such a DRC-covering of K_n and similarly we define $\rho_k(n)$ for the case when the cycle size is restricted to be k. We prove the following results:

Theorem A *When $n = 2p+1$, $\rho(n) = \frac{n^2-1}{8} = \frac{p(p+1)}{2}$. Furthermore, the DRC-covering of K_{2p+1} consists of p C_3 and $\frac{p(p-1)}{2}$ C_4.*

Theorem B *When $n = 2p$, $p \geq 2$, $\rho(n) = \left\lceil \frac{n^2+4}{8} \right\rceil = \left\lceil \frac{p^2+1}{2} \right\rceil$. Furthermore, when $n = 4q+4$, $q \geq 2$, the DRC-covering of K_{4q} consists of 4 C_3 and $2q^2-3$ C_4, and when $n = 4q+2$, $q \geq 1$, the DRC-covering of K_{4q+2} consists of 2 C_3 and $2q^2+2q-1$ C_4.*

Theorem C $\rho_4(2p+1) = \frac{p(p+1)}{2} + 1$ *, for $p \geq 3$; $\rho_4(4q) = 2q^2+1$, for $q \geq 2$, and $\rho_4(4q+2) = 2q^2+2q+2$, for $q \geq 1$.*

Let $\rho^*(n)$ be the minimum number of directed cycles needed in a DRC-covering of K_n^* and we also define $\rho_4^*(n)$ similarly. We have the following results.

Theorem D *When $n = 2p$, $\rho^*(n) = p^2$. Furthermore, we have a DRC-covering of K_{2p} with p $\vec{C}_2$ and p^2-p $\vec{C}_4$, and another covering with $2p$ $\vec{C}_3$ and p^2-2p $\vec{C}_4$.*

Theorem E *When $n = 2p+1$, $\rho^*(n) = p^2+p$. Furthermore, the DRC-covering of K_{2p+1} consists of $2p$ $\vec{C}_3$ and p^2-p $\vec{C}_4$.*

Theorem F $\rho_4^*(2p+1) = p^2+p+2$ *for $p \geq 3$, and $\rho_4^*(2p) = p^2+2$ for $p \geq 4$.*

Remark Theorem A and B were presented in SPAA conference [3].

2 Related results

Our problem has not yet been studied in the literature. However, without the disjoint routing constraint, there are some known results for finding the covering of K_n or K_n^* by cycles. Following four results determine the covering numbers when the cycle sizes are three and four.

Theorem 1 ([8, 9]) *The minimum number of 3-cycles required to cover the edges of K_n is $\left\lceil \frac{n}{3} \left\lceil \frac{n-1}{2} \right\rceil \right\rceil$.*

Theorem 2 ([2]) *The minimum number of 4-cycles required to cover the edges of K_n is*

$$\left\lceil \frac{n}{4} \left\lceil \frac{n-1}{2} \right\rceil \right\rceil + \epsilon(n) \quad \textit{with } \epsilon(n) = \begin{cases} 1 & \textit{if } n \equiv 3 \mod 8 \\ 0 & \textit{otherwise.} \end{cases}$$

Theorem 3 ([2]) *The minimum number of $\vec{C}_3$ required to cover the arcs of K_n^* is $\frac{n(n-1)}{3}$ if $n \equiv 0$ or $1 \mod 3$, except for $n = 6$ for which it is 12 and $\frac{n(n-1)+4}{3}$ if $n \equiv 2 \mod 3$.*

Theorem 4 ([2]) *The minimum number of $\vec{C}_4$ required to cover the arcs of K_n^* is $\left\lceil \frac{n(n-1)}{4} \right\rceil$, for $n > 4$.*

A related problem has been considered in the applications of optical networks in which one needs to find the minimum number of colors (wavelengths) required to color the edges of the logical graph I such that the paths associated to the requests with the same color are edge (arc) disjoint in the physical graph G (see [1]). In the undirected case for $G = C_n$, the problem is similar as finding a DRC cycle covering of K_n; indeed, we can associate to each cycle of a DRC-covering a color and conversely, we can build a cycle from the paths with the same color, as in C_n edge disjoint paths are also vertex disjoint. In [4], the answer was given for the undirected case of the problem. But unfortunately, the result was not correct for n even, and furthermore, the subnetworks corresponding to a given wavelength (color) are not cycles with small lengths. In the directed case, a minimum coloring was given in [11]. One of our results implies this coloring, but is stronger, as in [11] the paths with the same color are not vertex disjoint and again the subnetworks are not small directed cycles.

3 Our results

Let the vertices of the cycle C_n be labelled with integers modulo n, represented by the set $\{0, 1, ..., n-1\}$. A C_k satisfies DRC if and only if its vertices can be ordered cyclically modulo n, that is if the vertices can be written $(a_1, a_2, \ldots, a_k)$ with $0 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq n-1$. As an example, in Figure 2, the cycle $(0, 2, 3, 6, 0)$ satisfies DRC, but the cycle $(0, 4, 3, 6, 0)$ does not satisfy it.

We will first give the lower bounds for $\rho(n)$ and $\rho_4(n)$ and then construct the coverings which attain the lower bounds. We then extend the results to the directed cases.

3.1 Lower bounds

Proposition 5 $\rho(2p+1) \geq \frac{p(p+1)}{2}$ *with* $p \geq 1$, *and* $\rho(2p) \geq \frac{p^2+1}{2}$, $p \geq 2$.

Proof: Let C^j, $1 \leq j \leq \rho(n)$, be a cycle of a DRC-covering of K_n (remark that the cycles do not necessarily have the same length). The disjoint routing property implies that its

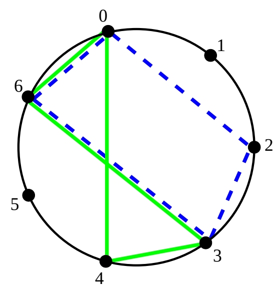

Figure 2: Disjoint Routing Constraint.

vertices are cyclically ordered modulo n. Thus C^j can be written $(a_1^j, a_2^j, \ldots, a_{k_j}^j, a_1^j)$, with $0 \leq a_1^j \leq a_2^j \leq \cdots \leq a_{k_j}^j \leq n-1$.

Let $\delta_i^j = a_{i+1}^j - a_i^j$, $1 \leq i \leq k_j - 1$, and $\delta_{k_j}^j = n + a_1^j - a_{k_j}^j$. The disjoint routing property implies $\sum_i \delta_i^j = n$.

For an edge $\{x, y\}$ of K_n with $x < y$, we call difference of the edge the value $y - x$ if $y - x \leq n/2$ or $x + n - y$ otherwise (it corresponds to the distance between x and y on a cycle of length n).

In the odd case, $n = 2p + 1$, the covering must contain the n edges of difference d for every d, $1 \leq d \leq p$. Each difference corresponds to a δ_i^j, with $\delta_i^j = d$ or $n - d$. Thus, $\sum_{i,j} \delta_i^j \geq \sum_{d=1}^{p} nd = n\frac{p(p+1)}{2}$. Remind that $\sum_i \delta_i^j = n$. Consequently, if the covering contains $\rho(n)$ cycles, we have $n\rho(n) \geq n\frac{p(p+1)}{2}$ and hence, $\rho(n) \geq \frac{p(p+1)}{2}$.

In the even case, $n = 2p$, the covering must contain n edges of difference d, where $1 \leq d \leq p - 1$, and $\frac{n}{2} = p$ edges of difference p. Furthermore, since the degree of the nodes in K_n is odd (equal to $n - 1$) and the degree of the nodes of a cycle is even (equal to 2), the covering must contain extra edges (i.e. in each vertex, there is an edge covered at least twice). Thus, there are at least $\frac{n}{2}$ extra edges of difference at least 1 in the covering. Consequently, $\sum_{i,j} \delta_i^j \geq \left(\sum_{d=1}^{p-1} nd\right) + pp + p = p(p^2 + 1)$ and if the covering contains $\rho(n)$ cycles, we obtain $n\rho(n) = 2p\rho(n) \geq p(p^2 + 1)$ and therefore, $\rho(n) \geq \frac{p^2+1}{2}$.

$\square$

Note that for both odd and even cases, the length of the cycles involved in the DRC-covering of K_n has no influence on the lower bound of $\rho(n)$. We show, in this paper, that the optimal solution can be obtained by using cycles of both length 3 and 4. Also, from Theorem 1, we can see that by using only cycles of length 3, it will not give the optimal solution.

Proposition 6 *$\rho_4(2p+1) \geq \frac{p(p+1)}{2} + 1$ with $p \geq 2$, and $\rho_4(4q) \geq 2q^2 + 1$ and $\rho_4(4q+2) \geq 2q^2 + 2q + 2$, with $q \geq 1$.*

Proof: In the odd case, $n = 2p+1$, we know by Proposition 5 that $\rho_4(2p+1) \geq \rho(2p+1) = \frac{p(p+1)}{2}$. If there was an equality, as $4\frac{p(p+1)}{2} = \frac{2p(2p+1)}{2} + p$, p extra edges are used because the covering is not a decomposition. So $\sum_{i,j} \delta_i^j \geq (2p+1)\frac{p(p+1)}{2} + p$, as the sum of the distances of the p extra edges is at least p. Therfore, $(2p+1)\rho_4(2p+1) \geq (2p+1)\frac{p(p+1)}{2} + p$ which implies $\rho_4(2p+1) \geq \frac{p(p+1)}{2} + 1$.

In the even case, for $n = 4q$, the bound is that of Proposition 5. For $n = 4q+2$, by Proposition 5, we have $\rho_4(4q+2) \geq \rho(4q+2) = 2q^2 + 2q + 1$. If $\rho_4(4q+2) = 2q^2 + 2q + 1$, then, as $4(2q^2 + 2q + 1) = \frac{(4q+2)(4q+1)}{2} + 2q + 3$, $2q + 3$ extra edges are used, and so $\sum_{i,j} \delta_i^j \geq (2q+1)^3 + 2q + 3$, and $(4q+2)\rho_4(4q+2) \geq (2q+1)^3 + 2q + 3$, which implies $\rho_4(4q+2) \geq 2q^2 + 2q + 1 + \frac{2}{4q+2} > 2q^2 + 2q + 1$.

□

3.2 Minimum DRC covering

Theorem A *When $n = 2p+1$, $\rho(n) = \frac{n^2-1}{8} = \frac{p(p+1)}{2}$. Furthermore, the DRC-covering of K_{2p+1} consists of p C_3 and $\frac{p(p-1)}{2}$ C_4.*

Proof: (By induction on p) K_3 is covered using one C_3. Thus, the theorem is true when $p = 1$.

Suppose now that the theorem is true for K_{2p+1}. We will show that it is also true for $n = 2p+3$. For that, let us arrange the vertices of K_{2p+3} in the following order: $A, 0, 1, \ldots, p-1, B, p, \ldots, 2p$.

We build a DRC-covering of K_{2p+3} from a DRC-covering of K_{2p+1} as follows. The cycles of the DRC-covering of K_{2p+3} will be

- the $p(p+1)/2$ cycles of a DRC-covering of the K_{2p+1} on vertices $0, 1, \ldots, p-1, p, \ldots, 2p$,
- the p C_4's of a DRC-decomposition of the $K_{2p,2}$ constructed between vertices $0, \ldots, p-1, p+1, \ldots, 2p$ on one side and vertices A and B on the other side, namely $(A, p-1-i, B, p+1+i, A)$, $0 \leq i \leq p-1$. Note that these cycles satisfy DRC.
- the C_3 (A, B, p, A).

One can check that each edge of K_{2p+3} is covered by exactly one of these cycles and altogether we have $p(p+1)/2 + p + 1 = (p+1)(p+2)/2$ cycles. Furthermore, there are exactly $p+1$ C_3's and $p(p+1)/2$ C_4's.

□

In Figure 3, we show a covering of K_5 obtained in that way. Let us called the vertices of K_5 $A, 0, B, 1, 2$ in that order. The DRC-covering of K_5 consists of the unique cycle $(0, 1, 2, 0)$ of the covering of K_3, plus the C_4 $(A, 0, B, 2, A)$ of a DRC-decomposition of the $K_{2,2}$ constructed between vertices A and B and vertices 0 and 2, plus the C_3 $(A, B, 1, A)$.

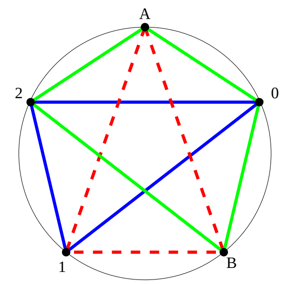

Figure 3: Covering of K_5 obtained from the covering of K_3.

Theorem B *When $n = 2p$, $p \geq 2$, $\rho(n) = \left\lceil \frac{n^2+4}{8} \right\rceil = \left\lceil \frac{p^2+1}{2} \right\rceil$. Furthermore, when $n = 4q+4$, $q \geq 2$, the DRC-covering of K_{4q} consists of 4 C_3 and $2q^2-3$ C_4, and when $n = 4q+2$, $q \geq 1$, the DRC-covering of K_{4q+2} consists of 2 C_3 and $2q^2+2q-1$ C_4.*

A covering of K_4 with one C_3 and 2 C_4 is given in Fig 1.

In order to prove this theorem, we first need to prove some lemmas.

Lemma 7 *K_6 can be covered by 2 C_3 and 3 C_4.*

Proof: The covering is given by the two C_3: $(0,1,3,0)$ and $(0,1,4,0)$, plus three C_4: $(0,2,4,5,0)$, $(1,2,3,5,1)$ and $(2,3,4,5,2)$, as shown in Figure 4. Furthermore, there are three edges, $\{0,1\}$, $\{2,3\}$ and $\{4,5\}$, covered exactly twice (they form a perfect matching). $\square$

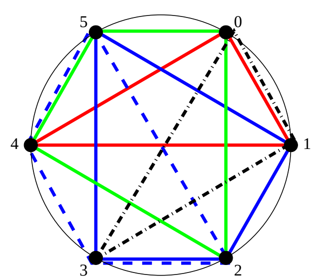

Figure 4: K_6

Lemma 8 *If there exists a DRC-covering of K_{4q+2} with $\rho(4q+2) = 2q^2+2q+1$ cycles, then there exists a DRC-covering of K_{4q+4} with $\rho(4q+4) = 2q^2+4q+3$ cycles.*

Proof: Let the vertices of K_{4q+4} be $A, 0, 1, \ldots, 2q, B, 2q+1, \ldots, 4q+1$ and arrange them in this order.

We build a DRC-covering of K_{4q+4} from a DRC-covering of K_{4q+2} as follows. The cycles of the DRC-covering of K_{4q+4} will be

- the $2q^2 + 2q + 1$ cycles of a DRC-covering of the K_{4q+2} on vertices $0, 1, \ldots, 2q, 2q+1, \ldots, 4q+1$,
- the $2q$ C_4's of a DRC-decomposition of the $K_{4q,2}$ constructed on vertices $1, \ldots, 2q, 2q+1, \ldots, 4q$ on one side, and vertices A and B on the other side, namely $(A, i, B, 2q+i, A)$, $1 \le i \le 2q$,
- the 2 triangles $(A, 0, B, A)$ and $(A, B, 4q+1, A)$.

One can check that every edge of K_{4q+4} is covered by one of these cycles and that altogether we have $2q^2 + 2q + 1 + 2q + 2 = 2q^2 + 4q + 3 = \left\lceil \frac{(2q+2)^2+1}{2} \right\rceil$ cycles. Furthermore, there are exactly 4 C_3's in the covering (2 from the DRC-covering of K_{4q+2} and the 2 extra C_3's).

□

To illustrate this proof, we indicate in Figure 5 the cycles involved in the covering of K_8.

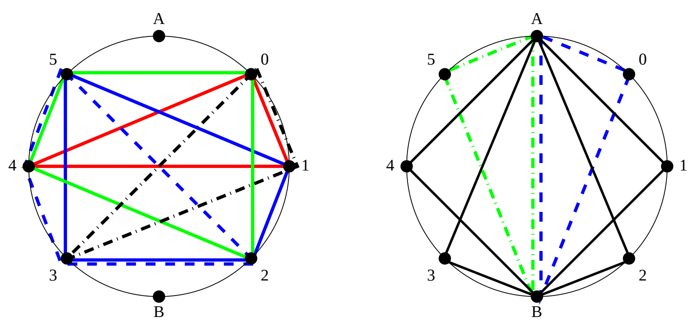

Figure 5: Cycles involved in the covering of K_8.

Lemma 9 *If there exists a DRC-covering of K_{4q+2} with $\rho(4q+2) = 2q^2 + 2q + 1$ cycles, then there exists a DRC-covering of K_{4q+6} with $\rho(4q+6) = 2q^2 + 6q + 5$ cycles.*

Proof: We will prove a stronger lemma, imposing some extra properties in the decomposition which will be kept in the construction.

Let us suppose that there exists a DRC-covering of K_{4q+2}, where the nodes $0, 1, \ldots, 4q+1$ are cyclically ordered, with the following properties:

- the edges (of the perfect matching) $(0,1),(2,3),\ldots,(4q,4q+1)$ are covered exactly twice, while other edges are covered exactly once.
- the edge $\{0,1\}$ belongs to the C_3 $(0,1,x,0)$, for some x different from $0,1$.

We will show that there exists a DRC-covering of K_{4q+6} with the same properties. Note that these properties are satisfied by the covering of K_6 of Lemma 7 (with x being either 3 or 4).
Let the vertices of K_{4q+6} be $0,A,B,1,\ldots,2q+1,C,D,2q+2,\ldots,4q+1$ and arrange them in this order. The cycles of the DRC-covering of K_{4q+6} will be

- the $2q^2+2q$ cycles of the covering of K_{4q+2} except the C_3 $(0,1,x,0)$,
- the $2q$ C_4's $(A,i,C,f(i),A)$, with $2 \leq i \leq 2q+1$ and where f is a bijection from $\{2,3,\ldots,2q+1\}$ to $\{2q+2,\ldots,4q+1\}$,
- the $2q+1$ C_4's $(B,j,D,g(j),B)$, $1 \leq j \leq 2q+1$, and where g is a bijection from $\{1,2,\ldots 2q+1\}$ to $\{2q+2,\ldots,4q+1,0\}$.
- the 3 C_4's (A,B,C,D,A), $(0,A,1,x,0)$, $(B,1,C,D,B)$ and the C_3 $(0,A,C,0)$,

One can check that each edge of K_{4q+6} is covered by one of these cycles and that altogether, we have $2q^2+2q+2q+2q+1+3+1 = 2q^2+6q+5 = \left\lceil \frac{(2q+2)^2+1}{2} \right\rceil$ cycles. Furthermore, there are still exactly 2 C_3's in the covering. Also, the edges $\{0,A\},\{B,1\},\{2,3\},\ldots,$ $\{2q,2q+1\},\ldots,\{C,D\},\{2q+2,2q+3\},\ldots,\{4q,4q+1\}$ (corresponding to a perfect matching) are covered twice, while other edges are covered only once. Moreover, the edge $\{0,A\}$, which is covered twice, appears in the C_3 $(0,A,C,0)$. Let us relabel the vertices of K_{4q+6} as follows: A (resp. B) becomes 1 (resp. 2), i, for $1 \leq i \leq 2q+1$, becomes $i+2$, C (resp. D) becomes $2q+4$ (resp. $2q+5$), and j, $2q+2 \leq j \leq 4q+1$ becomes $j+4$. Therefore, K_{4q+6} satisfy the induction properties.

□

Remark For the proof of Theorem C, we note that the two C_3's of the decomposition of K_6 are $(0,1,3,0)$ and $(0,1,4,0)$. So assuming that $x=3$ and after $q-1$ steps of induction, the two C_3's of K_{4q+2}, if $q \geq 2$, become $(0,1,2q+2,0)$ and $(0,2q-1,4q,0)$. Furthermore, we have a cycle $(1,i_0,2q+2,4q,1)$ (corresponding to the C_4 $(A,i,C,f(i),A)$ with $f(i_0)=4q$ in above construction).

Now, we are able to prove Theorem B.

Proof of Theorem B : (By induction)

The theorem is true for $n=6$ as shown by Lemma 7. Note that the covering of K_6 satisfies the two extra properties needed in the proof of Lemma 9. So using Lemma 9, one can build by induction the DRC-covering of K_{4q+2}, $q \geq 1$, by $\rho(4q+2) = 2q^2+2q+1$ cycles. Then, using Lemma 8, one can build the DRC-covering of K_{4q+4}, $q \geq 1$, by $\rho(4q+4) = 2q^2+4q+3$ cycles.

So Theorem B is proved.

□

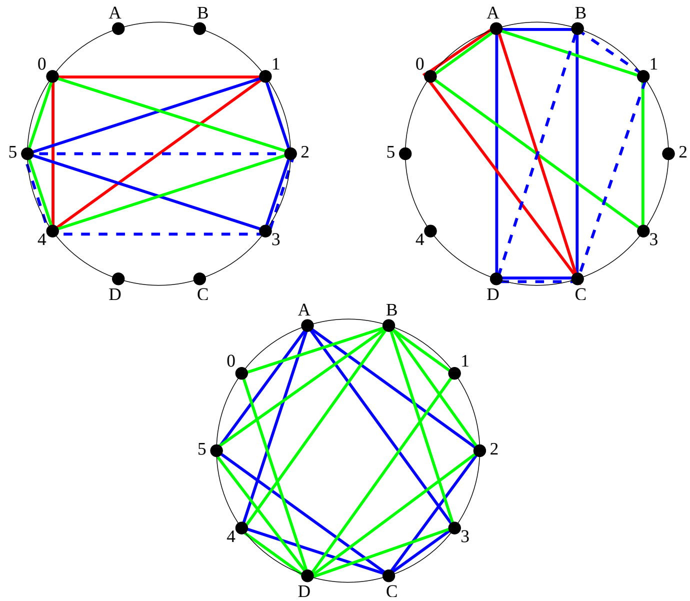

Figure 6: Cycles involved in the covering of K_{10}.

Theorem C $\rho_4(2p+1) = \frac{p(p+1)}{2} + 1$, *for* $p \geq 3$; $\rho_4(4q) = 2q^2 + 1$, *for* $q \geq 2$, *and* $\rho_4(4q+2) = 2q^2 + 2q + 2$, *for* $q \geq 1$.

Proof:

Case 1 : $n = 2p+1$**.** The proof is similar to that of Theorem A, and by induction, we will prove that there exists a C_4-covering of K_{2p+1} with vertices labelled $0, 1, \ldots, 2p$, containing a C_4 of the form $(0, p-1, p, p+1, 0)$ and the edge $\{p-1, p\}$ covered twice.

Note that it is true for $p = 3$ as we have a covering of K_7 with the 7 C_4's $(i, i+1, i+2, i+5, i)$, $0 \leq i \leq 6$, containing in particular the C_4 $(2, 3, 4, 0, 2)$ and with the edge $\{2, 3\}$ covered twice. Suppose that the induction hypothesis is true for K_{2p+1}. We will show that it is true for K_{2p+3}. Let the vertices of K_{2p+3} be $A, 0, 1, \ldots, p-1, B, p, \ldots, 2p$ and arrange them in this order. The C_4's of the DRC-covering of K_{2p+3} will be :

- the $\frac{p(p+1)}{2}$ C_4's of the covering of K_{2p+1}, except the C_4 $(0, p-1, p, p+1, 0)$,
- the $p-1$ C_4's of a DRC decomposition of $K_{2p-2,2}$, constructed between vertices $0, 1, \ldots, p-2, p+2, \ldots, 2p$ on one side, and the vertices A and B on the other side, namely the C_4's $(A, p-1-i, B, p+1+i, A)$, $1 \leq i \leq p-1$,
- the 3 C_4's $(0, p-1, B, p+1, 0)$, $(A, p-1, B, p, A)$ and $(A, B, p, p+1, A)$.

One can check that all the edges are covered and there are altogether $\frac{p(p+1)}{2}+p-1+3=\frac{(p+1)(p+2)}{2}+1$ C_4's. The covering contains the cycle $(A,p-1,B,p,A)$ which becomes, if we label the vertices of K_{2p+3} from 0 to $2p+2$, the cycle $(0,p,p+1,p+2,0)$; furthermore the edge $\{p-1,B\}$ which becomes $\{p,p+1\}$ after relabelling, is covered twice.

Case 2 : $n=4q+2$ **,** $q\geq 1$**.** Note that in the proof of Lemma 9, we obtain a covering of K_{4q+2} with $2q^2+2q-1$ C_4's and two C_3's which are (see remark) $(0,1,2q+2,0)$ and $(0,2q-1,4q,0)$. Delete these C_3's and consider the 3 C_4's $(0,1,2q+2,4q,0)$, $(0,2q-1,4q,4q+1,0)$ and $(0,\alpha,\beta,2q+2,0)$, with $0<\alpha<\beta<2q+2$. They contain all the edges of the deleted C_3's. Thus, we have a covering of K_{4q+2} with $2q^2+2q-1+3=2q^2+2q+2$ C_4's.

Case 3 : $n=4q+4$**.** We assume that $q>1$, and let us prove that $\rho_4(4q+4)=2q^2+4q+3$. Like in the proof of Lemma 8, let the vertices be $A,0,1,\ldots,2q,B,2q+1,\ldots,4q+1$. The proof of Lemma 8 also gives a decomposition of K_{4q+4} into the $2q^2+2q-1$ C_4's plus the 2 C_3's of the covering of K_{4q+2}, plus $2q+1$ C_4's of the decomposition of the $K_{4q+2,2}$ (with vertices $0,1,...,4q+1$ on one side and A,B on the other side), and plus the edge $\{A,B\}$. Using the remark of the proof of Lemma 9, we have a decomposition with $2q^2+4q$ C_4's plus the edge $\{A,B\}$, plus the C_3's $(0,1,2q+2,0)$ and $(0,2q-1,4q,0)$, and we know that one C_4 is of the form $(1,i_0,2q+2,4q,1)$. Let us replace the edge $\{A,B\}$, the 2 C_3's and this C_4 by the following 4 C_4's which cover all their edges: $(A,0,2q-1,B,A)$, $(0,1,2q+2,4q,0)$, $(0,1,i_0,2q+2,0)$ and $(1,\alpha,2q-1,4q,1)$, for some $1<\alpha<2q-1$, for example $\alpha=2$ (which supposes $q>1$). Hence, if $q\geq 2$, we have a covering with $2q^2+4q+3$ cycles.

Finally, if $q=1$ a covering of K_8 is given by the following 9 C_4's : $(i,i+1,i+4,i+5,i)$, $0\leq i\leq 3$, $(0,1,2,6,0)$, $(1,2,4,5,1)$, $(0,2,3,4,0)$, $(3,4,6,7,3)$ and $(1,3,5,7,1)$. Note that the edges $\{0,1\}$ and $\{4,5\}$ are covered twice.

□

Remark For $n=5$, $\rho_4(5)=5$, as in any C_4, there is exactly one edge $\{i,j\}$ such that $\delta_i^j=2$ since the only way to have $\sum_i \delta_i^j=5$ is to have 3 edges such that $\delta_i^j=1$ and one with $\delta_i^j=2$. Therefore, $\rho_4(5)\geq 5$. Finally a covering of K_5 is given by the 5 C_4's: $(i,i+1,i+2,i+4,i), 0\leq i\leq 4$.

4 Directed Case

We will now denote by $\rho^*(n)$ the minimum number of directed cycles needed in a DRC-covering of K_n^*.

Theorem D *When $n=2p$, $\rho^*(n)=p^2$. Furthermore, we have a DRC-covering of K_{2p}^* with p $\vec{C}_2$ and p^2-p $\vec{C}_4$, and another covering with $2p$ $\vec{C}_3$ and p^2-2p $\vec{C}_4$.*

Proof: Using a similar proof to that of Proposition 5, one can check that the lower bound is $\rho^*(2p)\geq p^2$ (we have no more the degree condition).

We will prove Theorem D by induction on p. First, a DRC-covering of K_4^* is given by the 4 $\vec{C}_3$ $(0,1,2,0)$, $(0,2,3,0)$, $(0,3,1,0)$ and $(1,3,2,1)$, and another is given by the 2 $\vec{C}_4$ $(0,1,2,3,0)$ and $(0,3,2,1,0)$ plus the 2 $\vec{C}_2$ $(0,2,0)$ and $(1,3,1)$.

Now, suppose that $\rho^*(2p) = p^2$ and let us prove that $\rho^*(2p+2) = (p+1)^2$. Let the vertices of K_{2p+2}^* be $A, 0, 1, \dots, p-1, B, p, \dots, 2p-1$ and arrange them in this order. A directed DRC-covering is given by the p^2 cycles of the covering of K_{2p}^*, plus the $2p$ $\vec{C}_4$ $(A, i, B, p+i, A)$ and $(A, p+i, B, i, A)$, $0 \le i \le p-1$, and plus the $\vec{C}_2$ (A, B, A). Thus, we have a covering of K_{2p+2}^* with $p^2+2p+1 = (p+1)^2$ cycles. If the p^2 cycles of the covering of K_{2p}^* are p $\vec{C}_2$ and $p^2 - p$ $\vec{C}_4$, we obtain a covering of K_{2q+2}^* with $p+1$ $\vec{C}_2$ and p^2+p $\vec{C}_4$.

Furthermore, one can replace the $\vec{C}_4$ $(A, 0, B, p, A)$ and the $\vec{C}_2$ (A, B, A) by the 2 $\vec{C}_3$ $(A, 0, B, A)$ and (B, p, A, B) to obtain a covering without $\vec{C}_2$, if we start from a covering of K_{2p}^* without $\vec{C}_2$.

□

Theorem E *When $n = 2p+1$, $\rho^*(n) = p^2 + p$. Furthermore, the DRC-covering of K_{2p+1}^* consists of $2p$ $\vec{C}_3$ and $p^2 - p$ $\vec{C}_4$.*

Proof: A proof similar to that of Theorem A gives $\rho^*(n) \ge p^2+p$. Given the DRC-covering of K_{2p+1}, obtained in Theorem A using p C_3's and $\frac{p^2-p}{2}$ C_4's, we replace each C_4 by two opposite $\vec{C}_4$ and each C_3 by two opposite $\vec{C}_3$. Thus, we obtain a DRC-covering using exactly p^2+p directed cycles, consisting of $2p$ $\vec{C}_3$ and $p^2 - p$ $\vec{C}_4$.

□

Theorem F *$\rho_4^*(2p+1) = p^2+p+2$ for $p \ge 3$, and $\rho_4^*(2p) = p^2+2$ for $p \ge 4$.*

Proof: **Case 1:** $n = 2p+1$. Suppose that $\rho_4^*(n) = p^2+p+1$. Then, as $4(p^2+p+1) = 2p(2p+1)+2p+4$, $2p+4$ extra edges are used and so $\sum_{i,j} \delta_i^j \ge (2p+1)p(p+1)+2p+4$, which implies $\rho_4^*(n) \ge p(p+1) + \frac{2p+4}{2p+1} > p^2+p+1$ (recall that the sum of the distances over each cycle is $2p+1$).

A covering with p^2+p+2 $\vec{C}_4$'s can be deduced from that of Theorem C by replacing each C_4 of this covering by two opposite $\vec{C}_4$.

Case 2 : $n = 2p$. Suppose that $\rho_4^*(n) = p^2+1$. Then, as $4(p^2+1) = 2p(2p-1)+2p+4$, $2p+4$ extra edges are used and so $\sum_{i,j} \delta_i^j \ge 2p(p-1)p + 2p^2 + 2p + 4$, which implies $\rho_4^*(n) \ge p^2 + \frac{2p+4}{2p} > p^2+1$.

A covering of K_8^* by 18 $\vec{C}_4$'s is deduced from that of K_8 by 9 C_4's, given in the proof of Theorem C, by replacing each C_4 of this covering by two opposite $\vec{C}_4$. Note that this covering contains the $\vec{C}_4$ $(0,1,4,5,0)$, which is of the form $(0,1,p,p+1,0)$, and that the arcs $(0,1)$ and $(4,5)$ are covered twice.

We will prove by induction that there exists a covering of K_{2p}^* with p^2+2 directed cycles containing the $\vec{C}_4$ $(0,1,p,p+1,0)$ and with the arcs $(0,1)$ and $(p,p+1)$ covered twice. That is true for $2p = 8$ as we have seen before. Suppose that it is true for K_{2p}^* and let the vertices of K^*_{2p+2} be $0, A, 1, \ldots, p, B, p+1, \ldots, 2p-1$ and arrange them in this order. A $\vec{C}_4$'s DRC-covering of K^*_{2p+2} is given by the p^2+1 cycles of the covering of K^*_{2p} excluding the $\vec{C}_4$ $(0,1,p,p+1,0)$, plus the $2p$ $\vec{C}_4$ $(A,i,B,p+i,A)$ and $(A,p+i,B,i,A)$, $1 \leq i \leq p$, and plus the 2 $\vec{C}_4$ $(A,1,p,B,A)$ and $(B,p+1,0,A,B)$. Thus, we have a covering of K^*_{2p+2} with $p^2+1+2p+2 = p^2+2p+3 = (p+1)^2+2$ cycles. Furthermore, this covering contains the cycle $(0,A,B,p+1,0)$ which becomes after relabelling $(0,1,p+1,p+2,0)$, and the arcs $(0,A)$ and (B,p) are covered twice.

□

Note that the above covering of K^*_n also gives an optimal covering of $2K_n$ as the lower bound was derived using $2K_n$.

5 Conclusion

The problem of the design of a survivable WDM network was considered as an extension of the classical edge covering problem by addition of the disjoint routing constraint. In particular, we have studied the case of a physical ring network with the all-to-all (K_n) communication instance. For this design problem, we give a solution with the optimal number of cycles as sub-networks.

Recently, in [5], we have extended the results to G being a torus (instead of C_n). It will be interesting to look at other WDM networks like tree of rings and grids. One can also consider other communications instances like λK_n, $\lambda K_{m,n}$ or general logical graphs.

Acknowledgment

The authors thank L. Chacon and F. Tillerot of France Telecom R & D for stimulating discussions. The work of the first two authors was partially supported by the french RNRT program (under the PORTO contract). The last author would like to thank the support of CNRS-INRIA-UNSA and the hospitality of the MASCOTTE project where the research was done during his visit.

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http://www.inria.fr
ISSN 0249-6399